CONCERNING FUNDAMENTAL GROUPS OF LOCALLY CONNECTED SUBSETS OF THE PLANE

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ABSTRACT. We show that every homomorphism from a one-dimensional Peano continuum to a planar Peano continuum is induced by a continuous map up to conjugation. We then prove that the topological structure of the space of points at which a planar Peano continuum is not semi-locally simply connected is determined solely by algebraic information contained in its fundamental group. Furthermore, we demonstrate how to reconstruct the topological structure of the space of points at which a planar Peano continuum is not semi-locally simply connected using only the subgroup lattice of its fundamental group.

INTRODUCTION

The Hawaiian earring is the one-point compactification of countably many disjoint arcs. Its fundamental group, which we will denote by $\mathbb{H}$, is called the Hawaiian earring group.

The fundamental groups of planar and one-dimensional Peano continua have been studied for quite some time. In [H], G. Higman studies the inverse limit of finite rank free groups which he calls the unrestricted free product of countably many copies of $\mathbb{Z}$. There he proves that this group is not free and that each of its free quotients has finite rank. He considers a subgroup P of the unrestricted product which turns out to be a Hawaiian earring group but he does not prove it there. In [CF], Curtis and Fort show that the Menger curve has fundamental group which embeds in the inverse limit of finite rank free groups. As well in [MM], Morgan and Morrison study the Hawaiian earring group. Cannon and Conner in [CC1]and [CC2] studied the fundamental groups of one-dimensional spaces. In the recent paper [ADTW], Akiyama, Dorfer, Thuswaldner, and Winkler studied the fundamental group of the Sierpinski-Gasket.

Katsuya Eda [E1] was the first to prove results concerning homomorphisms between the fundamental groups of one-dimensional Peano continua. He proved that every homomorphism from $\mathbb{H}$ to $\mathbb{H}$ is conjugate to a homomorphism which is induced by a continuous map. Summers gave a combinatorial proof of the same result in [S]. In [E2], Eda was able to extend his original result to show that a homomorphism from $\mathbb{H}$ to the fundamental

group of a one-dimensional Peano continuum is conjugate to a homomorphism which is induced by a continuous map.

In Section 2, we prove that homomorphisms from $\mathbb{H}$ to the fundamental group of a planar Peano continuum are conjugate to homomorphisms induced by continuous maps. This is then used to prove the following result.

Theorem 3.3 *Let X be a one-dimensional Peano continuum and Y a planar Peano continuum. Then for every homomorphism $\varphi : \pi_1(X,x) \to \pi_1(Y,y_0)$ then there exists a path $T : (I,0,1) \to (Y,y_0,y)$ and a continuous function $f : X \to Y$ such that $\hat{T} \circ \varphi = f_*$.*

Eda [E3] proved the analogue of Theorem 3.3 when both X and Y are one-dimensional Peano continua. In Section 3, we will show the necessary modifications of his proof to be able to extend it to the case when Y is planar.

Conjecture 0.1. *Theorem 3.3 is still true if X is allowed to be planar instead of one-dimensional.*

In [E2], Eda showed that the fundamental group of a one-dimensional continuum which is not semilocally simply connected at any point determines the homeomorphism type of the space. Eda and Conner in [CE] that the fundamental group of a one-dimensional continuum which is not semilocally simply connected at any point can be used to reconstruct the original space with its topology. We prove the following analogue for planar Peano continua.

Theorem 4.17 *If X is a planar Peano continuum, the homeomorphism type of $B(X)$ is completely determined by $\pi_1(X,x_0)$, where $B(X)$ is the subspace of points at which X is not semilocally simply connected.*

The proof that we present in Section 4 is for a planar Peano continuum; however, with a few insignificant changes our proof works for both planar and one-dimensional Peano continua. Our proof shows that what is really necessary for a results of this type isn't the one-dimensionality of the spaces involved but the fact that essential loops are cannot be homotoped to much (see Lemma A4 for an example of what this means in the planar case). Thus types of results are possible for any class of spaces where essential loops have this property.

1. Definitions

The one-point compactification of a sequence of disjoint arcs can be realized in the plane as the union of circles centered at $(0,\frac{1}{n})$ with radius $\frac{1}{n}$. We will use $\mathbf{E}$ to denote this subspace of the plane and $\mathbf{a}_n$ to denote the circle centered at $(0,\frac{1}{n})$ with radius $\frac{1}{n}$. Then $\pi_1(\mathbf{E},(0,0)) = \mathbb{H}$. At times it will be convenient to consider certain subspaces of $\mathbf{E}$ and certain subgroups of $\mathbb{H}$. We will let $\mathbf{E}_n = \bigcup_{i \geq n} \mathbf{a}_i$, $\mathbf{E}_e = \bigcup_{i \text{ even}} \mathbf{a}_i$ and $\mathbf{E}_o = \bigcup_{i \text{ odd}} \mathbf{a}_i$. Then let $\mathbb{H}_n$, $\mathbb{H}_e$, and $\mathbb{H}_o$ be their respective fundamental groups which are all isomorphic to $\mathbb{H}$.

Cannon and Conner have shown that $\mathbb{H}$ is generated in the sense of infinite products by a countable set of generators corresponding to the circles $\mathbf{a}_n$, where an infinite product is legal if each $\mathbf{a}_n$ is transversed only finitely many times. For more details see [CC1],[CC2]. We will refer to this infinite generating set for the fundamental group of $\mathbf{E}$ as $\{a_n\}$, i.e. a_n represents the canonical path which transverses counterclockwise $\mathbf{a}_n$ one time. We will frequently denote the base point $(0,0)$ of $\mathbf{E}$ by just 0.

We will use $[g]$ to represent the homotopy class relative to endpoints of the path g. Then a continuous function f induces an homomorphism on fundamental groups which we will denote by f_*. A loop f *is a reparametrization of g* if there exists continuous maps $\theta_1, \theta_2 : [0,1] \to [0,1]$ such that θ_i is nondecreasing, $\theta_i(0) = 0$ and $\theta_i(1) = 1$ for $i = 1,2$, and $f \circ \theta_1 = g \circ \theta_2$. This is an equivalence relation and we will write $f = g$.

Suppose that $f_i : I \to X$ are a sequence of paths such that $f_i(1) = f_{i+1}(0)$. Then $f_1 * f_2$ will denote the standard concatenation of paths and $\prod_i f_i$ the infinite concatenation when it is defined. We use $\overline{f}_1(t)$ to denote the path $\overline{f}_1(t) = f_1(1-t)$. For a path $\alpha : I \to X$, we will use $\widehat{\alpha}$ to represent the standard change of base point isomorphism, $\widehat{\alpha}([g]) = [\overline{\alpha} * g * \alpha]$. Then this isomorphism has inverse $\widehat{\overline{\alpha}}$.

Let $\{\mathbf{b}_n\}$ be a null sequence of loops based at x_0 in some metric space X. Then any function $f : (\mathbf{E}, 0) \to (X, x)$ which sends $\mathbf{a}_n$ continuously to $\mathbf{b}_n$ and sends 0 to x is continuous. Often when defining functions from $\mathbf{E}$ to X, we will only be interested in the property that $f_*(a_n) = [\mathbf{b}_n]$. Therefore to simplify the construction of such functions; when defining a function f we will only state that f maps $\mathbf{a}_n$ to $\mathbf{b}_n$ (for every n) meaning that f maps $\mathbf{a}_n$ continuously to $\mathbf{b}_n$ and sends 0 to the base point of $\mathbf{b}_n$ such that $f_*(a_n) = [\mathbf{b}_n]$.

A Peano continuum is a compact, connected, locally path connected, metric space.

We say that a nonconstant loop $f : I \to (X, x_0)$ is *reducible* if there is an open arc $(x, y) \subset I$ such that $f(x) = f(y)$ and the resulting loop based at $f(x) = f(y)$ defined by $f|_{[x,y]}$ is a non-constant nullhomotopic loop. A loop f is *reduced* if it is a reparametrization of a loop which is not reducible. A constant loop is, by definition reduced.

We will use f_r to denote the path where every nullhomotopic subpath of f is replaced by a constant path. Then f_r is a reduced representative of $[f]$. James Cannon and Greg Conner proved the existence and uniqueness (up to reparametrization) of reduced representatives of path class for one-dimensional spaces in [CC2]. We will use $[\cdot]_r$ to represent a reduced representative of the path class of $[\cdot]$.

Definition 1.1. Let X be a one-dimensional space. Let $g : I \to X$ be a reduced representative for the path class $[g]$. Then we say that $a : I \to X$ is a *head* for g if there exists $b : I \to X$ such that $g = a * b$, where $a * b$ is a reduced path. We will write $a \xrightarrow{h} g$. Additionally, we say that $b : I \to X$

is a *tail* for g if there exist $c : I \to X$ such that $g = c * \overline{b}$, where $c * \overline{b}$ is a reduced path and $\overline{b}$ is the path b traversed backwards. We will write $b \xrightarrow{t} g$.

We say that $t : I \to X$ is a *head-tail* for a reduced path $g : I \to X$ if t is a head and a tail for g and will write $t \xrightarrow{h-t} g$.

Since g is a reduced path; the paths a, b, and c are necessarily reduced paths.

2. Homomorphisms from $\mathbb{H}$

We will use the following theorem for one-dimensional spaces.

Theorem 2.1 (Eda [E2])**.** *Let $\varphi : \mathbb{H} \to \pi_1(X, x_0)$ a homomorphism into the fundamental group of a one-dimensional Peano continuum X. Then there exists a continuous function $f : (\boldsymbol{E}, 0) \to (X, x)$ and a path $T : (I, 0, 1) \to (X, x_0, x)$, with the property that $f_* = \widehat{T} \circ \varphi$. Additionally; if the image of φ is uncountable, then T is unique up to homotopy rel endpoints.*

2.1. **Delineation.** To prove Theorem 2.16, we will use an upper semicontinuous decomposition of the planar Peano continuum to get a continuous map into a one-dimensional Peano continuum which is injective on fundamental groups. This then allows us to, in some degree, reduce the planar case to the one-dimensional case. If π_k is this decomposition map, we show that we can lift the path T such that $\widehat{T} \circ \pi_{k*} \circ \varphi$ is induced be a continuous map. Then for α a lift of T, we show that $\widehat{\alpha} \circ \varphi$ is induced by a continuous function.

Definition 2.2. Let k be a line in the plane and X a planar Peano continuum. Let $\pi_k : X \to X/G$ be a decomposition map where the nontrivial decomposition elements of G are the maximal line segments in X which are parallel to k.

We will use X_k to denote the decomposition space corresponding to π_k. Cannon and Conner have shown that this is actually an upper semicontinuous decomposition, that X_k is a one-dimensional Peano continuum, and that the induced homomorphism on fundamental groups is injective (Theorem 1.4 in [CC3]).

Lemma 2.3. *If $g : I \to X$ is a path and $\pi_k \circ g$ has reduced representative f then there exists $\tilde{g} : I \to X$ such that $\pi_k \circ \tilde{g} = f$ up to reparametrization.*

Proof. If $\pi_k \circ g$ is reduced, we are done. Otherwise there exists an interval $[c, d]$ such that $\pi_k \circ g\big|_{[c,d]}$ is a non-constant nullhomotopic loop. Then $\pi_k \circ g(c) = \pi_k \circ g(d)$ which implies that the line segment $\overline{g(c)g(d)}$ is in contained in X. The loop $g\big|_{[c,d]} * \overline{g(d)g(c)}$ maps to $\pi_k \circ g\big|_{[c,d]}$ and hence must be nullhomotopic since π_{k*} is injective. Therefore g is homotopic to g' where the subpath $g\big|_{[c,d]}$ is replaced by the line segment $\overline{g(c)g(d)}$.

Suppose that $\{J_i = (c_i, d_i)\}$ is the set maximal disjoint open subintervals of I such that $\pi_k \circ g|_{\overline{J_i}}$ is non-constant nullhomotopic loop. Let l_i be a parametrization of the line segment from $g(c_i)$ to $g(d_i)$. Let $\tilde{g}$ be the path g where each subpath $g|_{[c_i,d_i]}$ is replace by l_i. It is not hard to check that $\tilde{g}$ defines a continuous path such that $g(t) = \tilde{g}(t)$ for $t \notin \cup_i J_i$.

We need to show that g is homotopic to $\tilde{g}$. Since g is uniformly continuous, diameter of $\{g|_{[c_i,d_i]}\}$ must converge to zero.

Claim: There exists homotopies $H_i : I \times [c_i, d_i] \to X$ with the property that $H_i|_{\{0\}\times[c_i,d_i]} = g|_{[c_i,d_i]}$, $H_i|_{\{1\}\times[c_i,d_i]} = \overline{l_i}$, and $H_i(I \times [c_i, d_i]) \to 0$.

Then Lemma A1 would imply that g is homotopic to $\tilde{g}$. By our choice of $\{J_i\}$, $\pi_k \circ \tilde{g}$ is reduced. This would then complete the proof of the lemma.

The claim is actually just a corollary of Cannon and Conner's proof that π_{k*} is injective. They show that if $h : \mathbb{S}^1 \to X \subset \mathbb{R}^2$ maps to a null-homotopic loop under π_k, then h bounds a disk contained in the bounded component of $\mathbb{R}^2 - h(\mathbb{S}^1)$ (see [CC3], p. 60-65). Hence we may choose H_i such that $\operatorname{diam}(H_i(I \times [c_i, d_i])) = \operatorname{diam}(g|_{[c_i,d_i]} * l_i) = \operatorname{diam}(g|_{[c_i,d_i]})$.

$\square$

Definition 2.4. If g maps to a reduced path under π_k, we will say that g is reduced with respect to k or g is k-reduced. For any path g; if $\tilde{g}$ is obtained from g as in Lemma 2.3 then we will say $\tilde{g}$ is obtained by reducing g with respect to k.

2.2. **Weight Function.** Before we proceed, we need to introduce a weight function. This weight function is a discrete version of the oscillation function defined by Cannon, Conner, and Zastrow in [CCZ]. The discrete version was also used in [K], since it is preserved under nerve approximation.

2.2.1. *Weight function.* For a path $f : I \to X$ and U, V disjoint open subsets of X, let $r_f : f^{-1}(U \cup V) \to \{-1, 1\}$ by $r_f(b) = 1$ if $f(b) \in U$ and $r_f(b) = -1$ if $f(b) \in V$. Let $\overline{w}_U^V(f) = \sup \left(-\sum_i r_f(b_i) \cdot r_f(b_{i+1})\right)$ taken over all increasing countable subsets of $f^{-1}(U \cup V)$. For any collection consisting of 0 or 1 point, we will consider the sum to be 0.

If the image of two consecutive points in our countable subset of $f^{-1}(U \cup V)$ are contained in the same open set, then the sum would increase by deleting one. Thus the supremum is obtained by choosing an increasing sequence of points from $f^{-1}(U \cup V)$ whose image alternates between U and V. Therefore $\overline{w}$ counts the number of times that the image of f alternates between U and V. If f is continuous and U, V have disjoint closures, then its image is compact and can only alternate between sets with disjoint closures finitely many times. So the supremum is actually realized for some finite set of points. If $U' \subset U$ and $V' \subset V$, then $\overline{w}_{U'}^{V'}(f) \leq \overline{w}_U^V(f)$.

Definition 2.5. The *weight* of f with respect to subsets A and B of X with disjoint closures is $w_B^A(f) = \inf \overline{w}_U^V(f)$ taken over all possible separations U

and V of $\overline{A}$ and $\overline{B}$. If $[f]$ is a homotopy equivalence class of functions, then
$w_B^A([f]) = \inf\limits_{f \sim f'} \{w_B^A(f')\}$.

Suppose that $\theta : I \to I$ is nondecreasing function such that $\theta(0) = 0$ and
$\theta(1) = 1$. Then it is not hard to see that $\overline{w}_V^U(f) = \overline{w}_V^U(f \circ \theta)$. Thus $\overline{w}$ is
preserved under reparameterization.

If $f : I \to X$ is a map into a one-dimensional space and $f\big|_{\overline{J}_{i)}}$ is a null-
homotopic subloop. Then $\overline{w}_V^U(f) \geq \overline{w}_V^U(f')$ where f' is the path obtained
by replacing the subpath $f\big|_{\overline{J}_{i)}}$ of f by a constant subpath. Thus we obtain
$\overline{w}_V^U([g]) = \overline{w}_V^U(f)$.

The set $\{\overline{w}_V^U(f) \mid U, V \text{are a separation of } \overline{A}, \overline{B}\}$ is a subset of the natural
numbers and hence has a minimum. Thus we may chose an open separation
U, V such that $w_B^A(f) = \overline{w}_U^V(f)$. For continuous f, $f^{-1}(U \cup V)$ can be
partitioned into a finite collection of disjoint open sets, I_i, in I with a natural
ordering ($I_i \leq I_j$ if $x \leq y$ for all $x \in I_i$ and $y \in I_J$) such that $f(I_i) \subset U$
or $f(I_i) \subset V$. If for some i, $f(I_i) \subset U \backslash \overline{A}$ (or $f(I_i) \subset V \backslash \overline{B}$), then there
would exist an open set contained in U (or V) containing the $\overline{A}$ (or $\overline{B}$)
which did not intersect $f(I_i)$ and thus alternate fewer times. Therefore,
$f(I_i)$ must intersect $\overline{A}$ (or $\overline{B}$). So points which realize the weight can be
chosen in the closures of A and B. Thus there exists a finite increasing set
of points $\{b_i\}$ which can be chosen to have image in the closures of A and B
such that $w_B^A(f) = \overline{w}_U^V(f) = \sum\limits_i -r_f(b_i) \cdot r_f(b_{i+1})$. We will sometimes write
$w_B^A(f) = \sum\limits_i -r_f(b_i) \cdot r_f(b_{i+1})$. This implicitly implies a choice of U and V
to define r_f. However; if the points are chosen to have image in the closure
of A and B, $r_f(b_i)$ is the same for every choice of U and V. Thus we will
ignore this choice at times.

Lemma 2.6. *If g is an essential loop in a one-dimensional space, there exist
sets O', O'' with disjoint closures such that, for all r, $w_{O''}^{O'}([g]^r) \geq r$.*

Proof. We may assume g is a reduced path since the weight of the reduced
path is less than or equal to the weight of all paths in its homotopy class.

Since the set of head-tails for g has a natural ordering which is bounded
there exists a maximal head-tail, t for g where $g = t * f * \overline{t}$ such that $f * f$
is reduced. Hence $w_B^A([g]^r) \geq w_B^A([f]^r) = r(w_B^A([f]))$ for all A and B. Since
g is essential, f is essential. Hence there exists O' and O'' with disjoint
closures such that $w_{O''}^{O'}([f]) \neq 0$. Then $w_{O''}^{O'}([g]^r) \geq r(w_{O''}^{O'}([f])) \geq r$, for any
r. $\qquad\qquad\square$

For f and g as in the proof of Lemma 2.6, we will say that f is the *core*
of g.

Let $h : C \to D$ be a function. Then a subset C' of C is *h-saturated* if
$C' = h^{-1}(D')$ for some $D' \subset D$.

Lemma 2.7. *Let A and B be disjoint closed π_k-saturated sets and A_k,B_k their respective images under π_k. Then $w_B^A(g) = w_{B_k}^{A_k}(\pi_k \circ g)$.*

This follows directly from the fact that the weight can be realized by a finite set of points.

Lemma 2.8. *The delineation map, π_k, preserve weights of homotopy classes on disjoint closed saturated sets, i.e. $w_B^A([g]) = w_{B_k}^{A_k}([\pi_k \circ g])$.*

Proof. Let $\tilde{g}$ be the path homotopic to g such that $\pi_k \circ \tilde{g}$ is reduced. Then $w_{B_k}^{A_k}([\pi_k \circ g]) = w_{B_k}^{A_k}(\pi_k \circ \tilde{g}) = w_B^A(\tilde{g}) \geq w_B^A([g])$. For g' homotopic to g $w_B^A(g') = w_{B_k}^{A_k}(\pi_k \circ g') \geq w_{B_k}^{A_k}(\pi_k \circ \tilde{g}) = w_{B_k}^{A_k}([\pi_k \circ g])$. Then $w_B^A([g]) \geq w_{B_k}^{A_k}([\pi_k \circ g])$.

Thus $w_B^A([g]) = w_{B_k}^{A_k}([\pi_k \circ g])$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Lemma 2.8 implies that a necessary condition for g to be k-reduced is that it have minimal weight in its path class with respect to all disjoint half-planes A and B with boundaries parallel to k.

In fact this condition is also sufficient. Suppose g has minimal weight in its path class with respect to all subsets A and B with boundaries parallel to k. If $\pi_k \circ g$ is not reduced, then there exists $g(c)$ and $g(d)$ such that $\overline{g(c)g(d)}$ is in contained in X and $\pi_k \circ g\big|_{[c,d]}$ is null-homotopic but not constant. Then $g\big|_{[c,d]}$ must not be contained in the line segment $\overline{g(c)g(d)}$. However, then g is homotopic to $\tilde{g}$ where $g\big|_{[c,d]}$ is replace by $\overline{g(c)g(d)}$ and the weight of $\tilde{g}$ is strictly less than the weight of g for some disjoint half-planes with boundaries parallel to k.

This characterization of begin reduced implies that if $\tilde{g}$ is obtained by reducing g, a k-reduced path, with respect to l; then $\tilde{g}$ is (k,l)-reduced (reduced with respect to both k and l).

2.3. **Induced by a continuous map.** For $\varphi : \mathbb{H} \to \pi_1(X, x_0)$ a fixed homomorphism into the fundamental group of a planar Peano continuum and each line k in the plane, we will use T_k to denote the path such that $\widehat{T}_k \circ (\pi_{k*} \circ \varphi)$ is induced by a continuous map.

The key to being able to reduced the planar case to the one-dimensional case is the following proposition.

Proposition 2.9. *For k a line in the plane, there exists α_k a path in X such that $\pi_k(\alpha_k) = T_k$.*

To prove this proposition we will construct a single word $a \in \mathbb{H}$ such that $T_k \xrightarrow{t} (\pi_k \circ \varphi(a))_r$.

Lemma 2.10. *Let X be a one-dimensional space. Suppose $\alpha, \beta, \gamma : [0,1] \to X$ are reduced paths such that $\alpha(1) = \beta(0)$, $\beta(1) = \gamma(0)$, and $\beta * \gamma$ is a*

reduced path. If there exists A, B such that $w_B^A(\alpha) < w_B^A(\beta)$ then $(\alpha * \beta)_r * \gamma$ is a reduced path.

Proof. The only way $(\alpha * \beta)_r * \gamma$ might not be a reduced path is if $\alpha = \gamma' * \overline{\beta}$ for some non-constant path γ'. This would imply $w_B^A(\beta) \leq w_B^A(\alpha)$ for all A, B. $\qquad\square$

Lemma 2.11. *Let X be a one-dimensional space. Suppose $\alpha, \beta, \gamma : [0,1] \to X$ are reduced paths such that $\alpha(1) = \beta(0)$, $\beta(1) = \gamma(0)$, and $\alpha * \beta$ is a reduced path. If $\operatorname{diam}(\gamma) < \operatorname{diam}(\beta)$ then $\alpha * (\beta * \gamma)_r$ is a reduced path.*

The proof follows is similar to the proof of Lemma 2.10.

In Lemma 2.10, the condition that there exists A, B such that $w_B^A(\alpha) < w_B^A(\beta)$ can be replaced by $\operatorname{diam}(\alpha) < \operatorname{diam}(\beta)$. The analogous change for Lemma 2.11 is also valid.

Lemma 2.12. *Let Y be a one-dimensional Peano continuum. Let $f : \boldsymbol{E} \to Y$ be a continuous function enjoying the property that $\operatorname{im}(f_*)$ is uncountable and let T be any path from $f(0)$ to $y \in Y$. Then there exists $a \in \mathbb{H}$ such that no non-constant head of T is a tail for $f_*(a)_r$, i.e. $f_*(a)_r * T$ is reduced.*

This can be derived as a result of Lemma 6.1 in [E2]. However; the proof of Proposition 2.14 depends heavily on the construction of a, so we will present a proof here for completeness.

Proof. Let b_n and t_n be subpaths of $f_*(a_n)_r$ such that $f_*(a_n)_r = t_n * b_n * \overline{t}_n$ where b_n is the core of $f_*(a_n)_r$. We may assume that b_n is not nullhomotopic, by passing to a subsequence if necessary.

Since f is continuous, the paths t_n and $f_*(a_n)_r$ must both form null sequences. Then we may choose an increasing sequence i_n such that $\operatorname{diam}(t_{i_n})$ is decreasing and $\operatorname{diam}(f_*(a_n)_r) \leq \dfrac{\operatorname{diam}(t_{i_{n-1}})}{2}$.

There exists an r_n such that for some A_n and B_n, $w_{B_n}^{A_n}(T) < w_{B_n}^{A_n}(b_{i_n} * \cdots * b_{i_n})$, the concatenation of r_n copies of b_{i_n}.

We will inductively define integers s_n and homotopy classes c_n in $\mathbb{H}$. Suppose r_i and c_i are defined for $i < n$.

Again, there exists an s_n such that for some A_n' and B_n', $w_{B_n'}^{A_n'}(f_*(\prod\limits_{i=1}^{n-1}(c_i))_r * t_{i_n}) < w_{B_n'}^{A_n'}(b_{i_n} * \cdots * b_{i_n})$, the concatenation of s_n copies of b_{i_n}. Let $c_n = a_{i_n}^{r_n + s_n}$. Then $f_*(c_n)_r = t_n * b_n^{r_n + s_n} * \overline{t}_n$.

Let $a = \prod\limits_{n=1}^{\infty} c_n$ and note that this is a well-define homotopy class in $\mathbb{H}$.

Claim. *For all $m \geq 2$,*

$$f_*(a)_r = (t_{i_1} * b_{i_1}^{s_1})_r * \prod_{j=2}^{m}\left(b_{i_{j-1}}^{r_{j-1}} * (\overline{t}_{i_{j-1}} * t_{i_j} * b_{i_j}^{s_{i_j}})_r\right) * \left(b_{i_m}^{r_{i_m}} * \overline{t}_{i_m} * f_*\left(\prod_{i=m+1}^{\infty} c_i\right)\right)_r$$

where the equality implies that the right hand side is reduced.

Proof of Claim. We can see that

$$b_{i_m}^{s_{i_m}} * \left(b_{i_m}^{r_{i_m}} * \bar{t}_{i_m} * f_*\big(\prod_{i=m+1}^{\infty} c_i \big) \right)_r$$

is a reduced path by Lemma 2.11 (in the notation of Lemma 2.11, $\alpha = b_{i_m}^{s_{i_m}}$, $\beta = b_{i_m}^{r_{i_m}} * \bar{t}_{i_m}$, and $\gamma = f_*\big(\prod_{i=m+1}^{\infty} c_i \big)_r$.

This give us that

$$\left((t_{i_1} * b_{i_1}^{s_1})_r * \prod_{j=2}^{m} \big(b_{i_{j-1}}^{r_{j-1}} * (\bar{t}_{i_{j-1}} * t_{i_j} * b_{i_j}^{s_{i_j}})_r \big) \right)_r * \left(b_{i_m}^{r_{i_m}} * \bar{t}_{i_m} * f_*\big(\prod_{i=m+1}^{\infty} c_i \big) \right)_r$$

is a reduced path by applying Lemma 2.10 with

$$\alpha = \left((t_{i_1} * b_{i_1}^{s_1})_r * \prod_{j=2}^{m-1} \big(b_{i_{j-1}}^{r_{j-1}} * (\bar{t}_{i_{j-1}} * t_{i_j} * b_{i_j}^{s_{i_j}})_r \big) * \bar{t}_{i_{m-1}} * t_{i_m} \right)_r, \beta = b_{i_m}^{s_{i_m}}, \text{ and }$$

$$\gamma = \left(b_{i_m}^{r_{i_m}} * \bar{t}_{i_m} * f_*\big(\prod_{i=m+1}^{\infty} c_i \big) \right)_r.$$

Then the claim would hold if

$$(t_{i_1} * b_{i_1}^{s_1})_r * \prod_{j=2}^{m} \big(b_{i_{j-1}}^{r_{j-1}} * (\bar{t}_{i_{j-1}} * t_{i_j} * b_{i_j}^{s_{i_j}})_r \big)$$

was a reduced path.

To simplify our notation, let t_{i_0}, b_{i_0} be constant paths and $r_{i_0} = 0$. With this notation all we must show is that

$$\prod_{j=1}^{m} \big(b_{i_{j-1}}^{r_{j-1}} * (\bar{t}_{i_{j-1}} * t_{i_j} * b_{i_j}^{s_{i_j}})_r \big)$$

is a reduced path.

We will proceed by induction on m. The case $m = 1$ is trivial. Suppose that the path is reduced for m.

Then

$$\left(\prod_{j=1}^{m} \big(b_{i_{j-1}}^{r_{j-1}} * (\bar{t}_{i_{j-1}} * t_{i_j} * b_{i_j}^{s_{i_j}})_r \big) \right) * b_{i_m}^{r_{i_m}} * \bar{t}_{i_m}$$

is a reduced path by Lemma 2.10 with $\alpha = \Big(\prod_{j=1}^{m-1} \big(b_{i_{j-1}}^{r_{j-1}} * (\bar{t}_{i_{j-1}} * t_{i_j} * b_{i_j}^{s_{i_j}})_r \big) *$

$(\bar{t}_{i_{m-1}} * t_{i_m})_r \Big)_r$, $\beta = b_{i_m}^{s_{i_m}}$, and $\gamma = b_{i_m}^{r_{i_m}} * \bar{t}_{i_m}$). Here we are also using the

inductive hypothesis that $\prod_{j=1}^{m} \big(b_{i_{j-1}}^{r_{j-1}} * (\bar{t}_{i_{j-1}} * t_{i_j} * b_{i_j}^{s_{i_j}})_r \big)$ is a reduced path.

The we complete the proof of the claim by applying Lemma 2.11 one more

time with $\alpha = \prod_{j=1}^{m} \big(b_{i_{j-1}}^{r_{j-1}} * (\bar{t}_{i_{j-1}} * t_{i_j} * b_{i_j}^{s_{i_j}})_r \big) * b_{i_m}^{r_{i_m}}$, $\beta = \bar{t}_{i_m}$, and $\gamma = t_{i_2} * b_{i_2}^{s_{i_2}}$).

$\square$

Suppose that $T|_{[0,s]}$ is a non-constant tail of $f_*(a)_r$. Then we can choose m_0 sufficiently large such that $\text{diam}\left(\left(b_{i_{m_0-1}}^{r_{i_{m_0-1}}} * \bar{t}_{i_{m_0-1}} * f_*(\prod_{i=m_0}^{\infty} c_i))_r\right)_r\right) < \text{diam}(T|_{[0,s]})$. But then the claim for $m = m_0$ would imply that $b_{m_0}^{r_{m_0}}$ is a subpath of T which contradicts our choice of r_{m_0}.

$\square$

Corollary 2.13. *If $\widehat{T}_k \circ \pi_k \circ \varphi = f_*$, then T_k is a tail for $\pi_{k*} \circ \varphi(a)$ where a is chosen as in Lemma 2.12.*

Proof. Note that $(\pi_{k*} \circ \varphi(a))_r = (T_k * f_*(a) * \overline{T}_k)_r$. Then Lemma 2.10 and Proposition 2.12 imply that $(T_k * f_*(a))_r * \overline{T}_k$ is a reduced path.

$\square$

Then Proposition 2.9 follows from Corollary 2.13 and Lemma 2.3.

Proposition 2.14. *If k and l are non-parallel lines in the plane, then there exists a path α in X such that $\pi_k(\alpha) = T_k$ and $\pi_l(\alpha) = T_l$.*

Proof. Let f by the continuous map such that $f_* = \widehat{T}_k \circ (\pi_{k*} \circ \varphi)$ and g the continuous map such that $g_* = \widehat{T}_l \circ (\pi_{l*} \circ \varphi)$.

We can then construct a word a as in Proposition 2.12 with the additional requirement that r_n and s_n satisfy the necessary conditions from proof of the proposition for f and g simultaneously. The key point is that at each stage of the construction of a, we could chose r_n, s_n as large as we want.

This allows us to find a single element $a \in \mathbb{H}$ such T_k is at tail for $\pi_k \circ f_*(a)$ and T_l is a tail for $\pi_l \circ g_*(a)$.

Fix h, a (k, l)-reduced path representative of $\varphi(a)$. Let α_k be the subpath of h mapping to T_k and β_l the subpath of h mapping to T_l. By our choice of a, $\pi_k \circ \alpha_k$ is the maximal tail of $\pi_k \circ h = (\pi_{k*} \circ \varphi(a))_r$ such that no terminal segment of $\pi_k \circ \alpha_k$ is a tail for $(T_k * f_*(a))_r$. Since no terminal segment of $\pi_k \circ \beta_l$ is a tail for $(T_k * f_*(a))_r$, $\pi_k \circ \beta_l$ is a subpath of $\pi_k \circ \alpha_k$. A similiar argument applied to π_l shows that $\pi_l \circ \alpha_k$ is a subpath of $\pi_l \circ \beta_l$. Thus $\beta_l = \alpha_k$.

$\square$

We will now show that α is the path such that $\widehat{\alpha} \circ \varphi$ is induced by a continuous map.

Lemma 2.15. *Let $\alpha_k : I \to X$ be a path with the property that $\pi_k \circ \alpha_k = T_k$, up to reparametrization. Let U be a π_k-saturated neighborhood of $\pi_k^{-1}(T_k(1))$. For sufficiently large n, every element of $\widehat{\alpha} \circ \varphi(\mathbb{H}_n)$ has a representative in U.*

Proof. If g is a loop based at a point $y \in \pi_k^{-1}(x)$ and $w_{U^c}^{\pi_k^{-1}(x)}([g]) = 0$, then g has a representative in U.

Let U be a π_k-saturated neighborhood of $\pi_k^{-1}(T_k(1))$. Let U' by an open π_k-saturated neighborhood of $\pi_k^{-1}(T_k(1))$ with closure contained in the interior of U. We must show that, for some sufficiently large n, $w_B^A(\widehat{\alpha} \circ \varphi(b)) = 0$ for all $b \in \mathbb{H}_n$ where $A = \pi_k^{-1}(T_k(1))$ and $B = \overline{U'}^c$.

Since $\widehat{T}_k \circ \pi_{k*} \circ \varphi$ is induced by a continuous map, there exists an N such that, for all $n > N$, $\widehat{T}_k \circ \pi_{k*} \circ \varphi(H_n) \subset \pi_1(\pi_k(U'), T_k(1))$. Hence $w_{B_k}^{A_k}(\widehat{T}_k \circ \pi_{k*} \circ \varphi(b)) = 0$ for all $b \in \mathbb{H}_n$ where $n > N$.

For $b \in \mathbb{H}_n$, where $n > N$, let f be a k-reduced representative of $\widehat{\alpha} \circ \varphi(b)$. Then the $w_B^A(\widehat{\alpha} \circ \varphi(b)) \leq w_B^A(f) = w_{B_k}^{A_k}(\pi_k \circ f) = w_{B_k}^{A_k}(\widehat{T}_k \circ \pi_{k*} \circ \varphi(b)) = 0$.
$\square$

Theorem 2.16. *Let $\varphi : \mathbb{H} \to \pi_1(X, x_0)$ a homomorphism into the fundamental group of a planar Peano continuum X. Then there exists a continuous function $f : (\boldsymbol{E}, 0) \to (X, x)$ and a path $\alpha : (I, 0, 1) \to (X, x_0, x)$, such that $f_* = \widehat{\alpha} \circ \varphi$. Additionally, if the image of φ is uncountable the α is unique up to homotopy relative to its endpoints.*

Proof. For k and l nonparallel lines in the plane, there exists a path α in X such that $\pi_k(\alpha) = T_k$ and $\pi_l(\alpha) = T_l$, by Lemma 2.14.

It is sufficient to show that for any neighborhood U of $\alpha(1)$ there exists an N such that every element of $\widehat{\alpha} \circ \varphi(\mathbb{H}_n)$ has a representative in U.

This is done by finding U_l and U_k such that $U_k \cap U_l \subset U$ and U_l is a π_l-saturated neighborhood of $\pi_l^{-1} \circ \pi_l(\alpha(1))$ and U_k is a π_k-saturated neighborhood of $\pi_k^{-1} \circ \pi_k(\alpha(1))$. The uniqueness follows from the uniqueness of T_k and the injectivity of π_{k*}.
$\square$

3. Homomorphisms induced by continuous functions

Throughout this section, we will assume all paths are from the unit interval $[0, 1]$. We will use j for the inclusion map of subsets of X into X. When there is no confusion, we will allow the domain of j to change without comment.

Definition 3.1. Let $\varphi : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ be a homomorphism. We will say x is φ-bad if $\varphi \circ \widehat{\alpha} \circ j_*(\pi_1(B_\epsilon(x), x))$ is nontrivial for all $\epsilon > 0$ where α is a path from x_0 to x. Note this is independent of the chosen α. We will say that X is φ-bad, if it is at each point. We will write $B_\varphi(X)$ for the set of φ-bad points of X.

If $X = Y$ and $\varphi = \mathrm{id}$, then this definition is exactly the definition given in Section 4 for *bad* and we will generally drop φ from the notation $B_\varphi(X)$.

Cannon and Conner in [CC3] defined $B(X)$ for a certain class of subset of the plane which where homotopy equivalent to planar Peano continua. It is a trivial exercise to see that these definitions are compatible. Conner and Eda in [CE] defined a similar set for one-dimensional spaces which they referred to as the set of *wild* points. While both terms have appeared in the literature, here we will use the term bad.

Proposition 3.2. Let X and Y be one-dimensional or planar Peano continua. Let $\varphi : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ be a homomorphism . For every path $\alpha : (I, 0, 1) \to (X, x_0, x)$ such that $x \in B_\varphi(X)$, there exists a unique (up to homotopy rel endpoints) path $\mathfrak{a} : (I, 0, 1) \to (Y, y_0, y)$ where $\mathfrak{a}$ depends only on the homotopy class (rel endpoints) of α and y depends only on x which enjoys the property that for every continuous map $g : (\mathbf{E}, 0) \to (X, \alpha(1))$ there exists $h : (\mathbf{E}, 0) \to (Y, \mathfrak{a}(1))$ such that $\varphi \circ \widehat{\overline{\alpha}} \circ g_* = \widehat{\overline{\mathfrak{a}}} \circ h_*$.

If X and Y one-dimensional, then this is Lemma 5.1 in [E2]. The proof we present here is essentially that same as Eda's proof. We give the proof for the planar case here to establish notation and to show the insignificant changes required in the planar situation.

Proof. Since $x \in B_\varphi(X)$, there exists a sequence of loops, $\{\mathbf{b}_i\}$, based at x which converge to x with the property that $[\alpha * \mathbf{b}_i * \overline{\alpha}]$ (or the conjugation by any other path from x_0 to x) map non-trivially under φ. Then Theorem 2.16 implies that there exists a unique $\mathfrak{a} : (I, 0, 1) \to (Y, y_0, y)$ and an $h :$ $(\mathbf{E}, 0) \to (Y, y)$ such that $\varphi \circ \widehat{\overline{\alpha}} \circ g_* = \widehat{\overline{\mathfrak{a}}} \circ h_*$ where g is the continuous map which sends $\mathbf{a}_i$ to $\mathbf{b}_i$.

Suppose $g' : (\mathbf{E}, 0) \to (X, x)$ is any continuous function. Let $g'' : (\mathbf{E}, 0) \to (X, x)$ be the function which sends $\mathbf{E}_e$ to X by g and sends $\mathbf{E}_o$ to X by g'. Then g'' is continuous since $g(0) = g'(0) = x$. Then there exists $\mathfrak{c}$ and $h'' : (\mathbf{E}, 0) \to (Y, y)$ such at $\varphi \circ \widehat{\overline{\alpha}} \circ g''_* = \widehat{\overline{\mathfrak{c}}} \circ h''_*$.

Then $\widehat{\overline{\mathfrak{a}}} \circ h_* = \varphi \circ \widehat{\overline{\alpha}} \circ g_* = \varphi \circ \widehat{\overline{\alpha}} \circ (g''|_{\mathbf{E}_e})_* = \widehat{\overline{\mathfrak{c}}} \circ (h''|_{\mathbf{E}_e})_*$. Then the uniqueness in Theorem 2.16 implies that $\mathfrak{a}$ is homotopic to $\mathfrak{c}$. Hence $\varphi \circ \widehat{\overline{\alpha}} \circ g'_* = \varphi \circ \widehat{\overline{\alpha}} \circ (g''|_{\mathbf{E}_o})_* = \widehat{\overline{\mathfrak{a}}} \circ (h''|_{\mathbf{E}_o})_*$.

Now we must only show that y depends only on x. Suppose $\alpha' : (I, 0, 1) \to (X, x_0, x)$ is a path which is not necessarily homotopic to α. Then for some $\mathfrak{a}'$ and h', we have $\varphi \circ \widehat{\overline{\alpha'}} \circ g_* = \widehat{\overline{\mathfrak{a'}}} \circ h'_*$.

$$
\begin{aligned}
\widehat{\overline{\mathfrak{a'}}} \circ h'_*(u) &= \varphi \circ \widehat{\overline{\alpha'}} \circ g_*(u) \\
&= \varphi \circ \overline{\alpha' * \overline{\alpha}} \; \widehat{} \circ \widehat{\overline{\alpha}} \circ g_*(u) \\
&= \varphi(\alpha' * \overline{\alpha}) \cdot \varphi \circ \widehat{\overline{\alpha}} \circ g_*(u) \cdot (\varphi(\alpha' * \overline{\alpha}))^{-1} \\
&= \varphi(\alpha' * \overline{\alpha}) \cdot \widehat{\overline{\mathfrak{a}}} \circ h_*(u) \cdot \varphi^{-1}(\alpha' * \overline{\alpha})
\end{aligned}
$$

Hence $h'(\mathbf{a}_n)$ is freely homotopic to $h(\mathbf{a}_n)$ for all i.

If $\mathfrak{a}'(1) \neq \mathfrak{a}(1)$, then we can find n sufficiently large such that $h(\mathbf{a}_n)$ and $h'(\mathbf{a}_n)$ can be separated by disjoint balls in Y. This contradictions Lemma A4. (We can choose n sufficiently large such that the disjoint balls satisfy the hypothesis of Lemma A4.) $\square$

We will now use this theorem as a base to prove the following result.

Theorem 3.3. *Let $\varphi : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ be a homomorphism where X is a one-dimensional Peano continuum and Y is a planar Peano continua.*

Then there exists a path $T : (I, 0, 1) \to (Y, y_0, y)$ and a continuous function $f : X \to Y$ such that $\hat{T} \circ \varphi = f_*$.

When both X and Y are one-dimensional, this is Theorem 1.2 in Eda's recent paper [E3]. We will show how to adapt Eda's to the case where Y is a planar Peano continuum.

Remark 3.4. Eda's proof basically involved two steps. First, he showed that that Theorem 3.3 was true if $B(X) = B_\varphi(X)$. The second step was to show that given any homomorphism one could find a retraction $r : X \to X$ fixing $B_\varphi(x)$ such that $B(r(X)) = B_{\varphi \circ r_*}(r(X))$. Then he was able to just apply the special case proved in step one. His construction of this retraction did not utilize the fact that φ mapped into the fundamental group of a one-dimensional continuum. Hence, the same method works to construct a retraction of X with the necessary properties when φ maps into the fundamental group of a planar continuum.

Therefore to complete the proof of Theorem 3.3, it will be sufficient to show the theorem holds with the additional hypothesis that $B(X) = B_\varphi(X)$.

To simplify notation; we will use $[f]_r$ to represent the reduced representative of $[f]$ when f maps into a one-dimensional continuum, as well as, a (k, l)-reduced representative when f maps into a planar Peano continuum for some fix perpendicular lines k and l. *Hence, the reader should take extra care when reading the proofs and definitions to understand why the arguments work in each case.*

Theorem 3.5 (Meistrup, [CM]). *Every one-dimensional Peano continuum is homotopy equivalent to a one-dimensional Peano continuum X such that X is a finite connected graph or $X \backslash B(X)$ is null sequence of disjoint open arcs with end points in $B(X)$.*

Such a continuum is called *arc-reduced*.

We will fix X an arc-reduced one-dimensional Peano continua, Y a planar Peano continuum and $\varphi : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ a homomorphism of their fundamental groups such that $B(X) = B_\varphi(X)$. Since we are only concerned about φ up to composition with a change of base point isomorphism, we may assume that $x_0 \in B(X)$.

Proposition 3.2 gives us a function, ψ from the set of reduced paths with initial point x_0 and terminal point in $B_\varphi(X)$ to the set of paths in Y from y_0 to points in $B(Y)$ by $\psi(\alpha) = [\mathfrak{a}]_r$. We want to extended ψ to reduced paths from x_0 to arbitrary points of X. However, before we can do this we need to understand the relationship of φ with the good arcs in X.

Since X is arc-reduced, $X \backslash B(X)$ is a null sequence of disjoint open arcs, A_i. For each i, fix a homeomorphism $m_i : (0, 1) \to A_i$. Then extend m_i to $[0, 1]$ by sending $0, 1$ to the corresponding endpoints of A_i in $B(X)$.

Let $\mathfrak{m_i}$ be a (k, l)-reduced representative of $[\overline{\psi(\beta_i)} * \psi(\beta_i * m_i)]$ where β_i is a path from x_0 to $m_i(0)$. The next two lemmas will show that $\mathfrak{m_i}$ is independent of our chosen β_i.

The proof we present here for the next two lemmas is essentially that same as Eda's proofs of the corresponding lemmas. They are presented here to show how the hold in the planar case.

Lemma 3.6. *[Lemma 3.3, [E3]] Let α, β be paths from x_0 to $x \in B(X)$. Then $\varphi(\beta * \overline{\alpha}) = [\psi(\beta) * \overline{\psi(\alpha)}]$.*

Proof. For all continuous $h : (\mathbf{E}, 0) \to (X, x)$, there exists $g, g' : (\mathbf{E}, 0) \to (Y, y)$ such that $\varphi \circ \hat{\overline{\beta}} \circ h_* = \overline{\psi(\beta)} \circ g_*$ and $\varphi \circ \hat{\overline{\alpha}} \circ h_* = \overline{\psi(\alpha)} \circ g'_*$. Then for all $u \in \mathbb{H}$,

$$
\begin{aligned}
\overline{\psi(\hat{\alpha})} \circ g'_*(u) &= \varphi \circ \hat{\overline{\alpha}} \circ h_*(u) \\
&= \varphi \circ \overline{\alpha * \overline{\beta}} \circ \hat{\overline{\beta}} \circ h_*(u) \\
&= \varphi(\alpha * \overline{\beta}) \cdot \varphi \circ \hat{\overline{\beta}} \circ h_*(u) \cdot \left(\varphi(\alpha * \overline{\beta}) \right)^{-1} \\
&= \varphi(\alpha * \overline{\beta}) \cdot \overline{\psi(\hat{\beta})} \circ g_*(u) \cdot \left(\varphi(\alpha * \overline{\beta}) \right)^{-1} \\
&= \overline{\varphi(\alpha * \overline{\beta})_r * \psi(\hat{\beta})} \circ g_*(u)
\end{aligned}
$$

Then Theorem 2.16 implies that $\psi(\alpha)$ is homotopic to $\varphi(\alpha * \overline{\beta})_r * \psi(\beta)$ which completes the proof. $\square$

Lemma 3.7. *[Lemma 3.4, [E3]] The path $\mathfrak{m_i}$ is independent of the path β_i from x_0 to the initial point of m_i.*

Proof. Let β, γ be paths from x_0 to the initial point of m_i. Then by Lemma 3.6,

$$
\begin{aligned}
[\overline{\psi(\gamma)} * \psi(\gamma * m_i) * \overline{\psi(\beta * m_i)} * \psi(\beta)] &= [\overline{\psi(\gamma)}] \cdot \varphi([\gamma * m_i * \overline{m_i} * \overline{\beta}]) \cdot [\psi(\beta)] \\
&= [\overline{\psi(\gamma)}] \cdot \varphi([\gamma * \overline{\beta}]) \cdot [\psi(\beta)] \\
&= [\overline{\psi(\gamma)} * \psi(\gamma) * \psi(\overline{\beta}) * \psi(\beta)] = [x_0].
\end{aligned}
$$
$\square$

We can now extended ψ to reduced paths from x_0 to arbitrary points of X. Let $p : (I, 0, 1) \to (X, x_0, x)$ be a reduced path. If $x \in B(X)$, then $\psi(p)$ is already defined. Otherwise, $x = m_i(t_0)$ for some i and t_0. Then p, up to re-parametrization, is $p' * m_i|_{[0,t_0]}$ or $p' * \overline{m_i}|_{[0,1-t_0]}$ where p' is uniquely determined and is a path between points in $B(X)$. Then we will define $\psi(p)$ to be $\psi(p') * \mathfrak{m_i}|[0, t_0]$ if $p = p' * m_i|_{[0,t_0]}$ or $\psi(p') * \overline{\mathfrak{m_i}}|_{[0,1-t_0]}$ if $p = p' * \overline{m_i}|_{[0,1-t_0]}$.

Again, we will use X_k, X_l to denote the decomposition space corresponding to π_k, π_l respectively, as in Section 2.1.

Remark 3.8. The function ψ from reduced paths based at x_0 to (k, l)-reduced paths in Y depends on φ and our choice of $\{m_i\}$. By the same method, we can define a function ψ_k (or ψ_l) from reduced paths in X based at x_0 to reduced paths in X_k (or X_l) based at $\pi_k(x_0)$ (or $\pi_l(x_0)$) using the homomorphism $\pi_{k*} \circ \varphi$ (or $\pi_l \circ \varphi$) and the same homeomorphisms $\{m_i\}$ on

the arcs of X not in $B(X)$. A corollary of our proof of Theorem 2.16 is that $\pi_k \circ \psi = \psi_k$ (or $\pi_l \circ \psi = \psi_l$) up to a re-parametrization of $\{\mathfrak{m}_i\}$.

We are now ready to define our function between X and Y which will induce φ up to conjugation. Let $f : X \to Y$ by $f(x) = \psi(\alpha)(1)$ for a path α from x_0 to x. If $x \in B(X)$, this is independent of α by Proposition 3.2. If $x \in X \backslash B(X)$, this is independent of α by Lemma 3.7. To demonstrate the continuity of f, we will need the following lemma.

Lemma 3.9. *Let $\alpha : (I, 0, 1) \to (X, x_0, x)$ be a path and $p \in \{k, l\}$. Let $\alpha_t : I \to X$ be the path α restricted to $[0, t]$. Then for every t_0 and $\epsilon > 0$, there exists a $\delta > 0$ such that if $|t - t_0| < \delta$ then $\mathrm{diam}\left(\left(\overline{\psi_p(\alpha_t)} * \psi_p(\alpha_{t_0}) \right)_r \right) < \epsilon$.*

This is a consequence of Lemma 3.7 and Lemma 3.8 in [E3]. It is also equivalent to his claim that if t_n converges than the $\psi(\alpha_{t_n})$ converge in the tail-limit topology.

Let $T : (I, 0, 1) \to (Y, y_0, y)$ be the image of the constant path at x_0 under ψ. Then the claim is that $\widehat{T} \circ \varphi = f_*$.

Lemma 3.10. *If α is a loop in X based at x_0, then $\psi(\alpha)$ is homotopic rel endpoints to $\varphi([\alpha])_r * T$.*

Proof. Since X is not semilocally simple connected at x_0; we can find, as in the proof of Proposition 3.2, a continuous map $g : (\mathbf{E}, \mathbf{0}) \to (X, x_0)$ such that $\varphi \circ g_*$ has uncountable image. Then by Proposition 3.2, $\varphi \circ \hat{\bar{\alpha}} \circ g_* = \overline{\widehat{\psi(\alpha)}} \circ h_*$ where $h : (\mathbf{E}, \mathbf{0}) \to (Y, y)$ is continuous. Thus $\overline{\widehat{\varphi([\alpha])}_r} \circ \varphi \circ g_* = \overline{\widehat{\psi(\alpha)}} \circ h_*$. However, $\{ (\overline{T} * \varphi \circ g_*(a_i)_r * T)_r \}$ is a sequence of loops converging to y by our choice of T. Hence, the map $h' : (\mathbf{E}, \mathbf{0}) \to (Y, y)$ which sends ∂_i to $(\overline{T} * \varphi \circ g_*(a_i)_r * T)_r$ is continuous. Then we get $h'_* = \hat{T} \circ \varphi \circ g_*$. Therefore we can obtain $\overline{\widehat{\varphi([\alpha])_r * T}} \circ h'_* = \overline{\widehat{\varphi([\alpha])_r}} \circ \hat{\bar{T}} \circ h'_* = \overline{\widehat{\varphi([\alpha])_r}} \circ \varphi \circ h_* = \overline{\widehat{\psi(\alpha)}} \circ h_*$. $\square$

Lemma 3.11. *Let $\alpha : (I, 0, 1) \to (X, x_0, x)$ be a path and $p \in \{k, l\}$. Then $T * (f \circ \alpha)$ is homotopic rel endpoints to $\psi_p(\alpha)$.*

Proof. Let $\alpha : (I, 0, 1) \to (X, x_0, x)$ be a path. Let $\alpha_t : I \to X$ be the path α restricted to $[0, t]$.

We will now define a homotopy between $\psi_p(\alpha)$ and $T * (f \circ \alpha)$. Let $H : [0, 2] \times [0, 1] \to Y$ by $H_t|_{[0,1]} = \psi_p(\alpha_t)$ and $H_t|_{[1,2]}(s) = f \circ \alpha|_{[t,1]}((2 - s)t + (s - 1))$. Each horizontal t-slices is the path $\psi_p(\alpha_t) * f \circ \alpha|_{[t,1]}$. Clearly $H|_{[1,2] \times [0,1]}$ is continuous. Therefore to show that H is actually a homotopy (i.e. continuous), we need only show that our $\psi_p(\alpha_t)$ have been chosen so as to vary continuously which is implied by Lemma 3.9.

$\square$

Proposition 3.12. Y **is planar.** *Let $\alpha : (I, 0, 1) \to (X, x_0, x)$ be a path. Then $T * (f \circ \alpha)$ is homotopic rel endpoints to $\psi(\alpha)$.*

Proof. This follows immediately from the fact that $\pi_k \circ T$ is homotopic to $\psi_k(x_0)$ and the injectivity of π_{k*}.   □

Fix a loop α in X based at x_0. Then $T * f \circ \alpha$ is homotopic to $\psi(\alpha)$ by Proposition 3.12. Additionally, Lemma 3.10 implies that $\psi(\alpha)$ is homtopic to $\varphi([\alpha])_r * T$. Hence $T * f \circ \alpha * \overline{T}$ is homtopic to $\varphi([\alpha])_r$. Therefore $\hat{\overline{T}} \circ f_* = \varphi$ which completes the proof of Theorem 3.3.

4. RECOVERING THE BAD SET

Recall that Eda and Conner in [CE] showed that the fundamental group of a one-dimensional continuum which is not semilocally simply connected at any point can be used to reconstruct the original space. The construction that was used there is similar to the one we will present here for the planar case.

We will begin by defining the bad set.

Definition 4.1. We will denote the set of points at which X is not semilo-cally simply connected by $B(X)$, the *bad set of X*.

Lemma 4.2. *Let be X a planar Peano continuum. Then $B(X)$ is fixed by every self-homotopy of X.*

Proof. Let $f : X \to X$ be homotopic to the identity on X. Then any loop is homotopic to its image under f. Suppose that $x \in B(X)$ such that $f(x) \neq x$. Since X is not semilocally simple connected at x, there exists a closed disk neighborhood D of x in the plane with the property that ∂D is not contained in X and $D \cap f(D) = \emptyset$. Then any loop contained in D can be homotoped into $f(D)$. However; this is a contradiction of Lemma A4, since there exist essential loops contained in the interior of $D \cap X$.

□

Definition 4.3. We will say that H is a *pseudo-Hawaiian earring subgroup of G*, if H is a subgroup of G and the uncountable homomorphic image of $\mathbb{H}$.

It will sometimes be convenient to talk about a generating set, in the sense of infinite products, for a (pseudo-)Hawaiian earring subgroup of $\pi_1(X, x_0)$. For each (pseudo-)Hawaiian earring subgroup $H \in \pi_1(X, x_0)$, we may choose an (homomorphism) isomorphism $\varphi : \mathbb{H} \to H$. We will say that $\{\varphi(a_i)\}$ *generates (in the sense of infinite products) H* and will write $H = \langle\langle \varphi(a_n) \rangle\rangle$.

In this section, we will show that there exists a natural equivalence rela-tion on pseudo-Hawaiian earring subgroups. Natural in the sense that it is determined by the topology and determines the topology of $B(X)$.

A natural question is what are the possible isomorphism types of pseudo-Hawaiian earring subgroups. Conner has conjectured that every uncountable homomorphic image of an Hawaiian earring group which embeds in the fundamental group of a planar or one-dimensional Peano continuum is a Hawaiian earring group.

4.1. **Equivalence classes of pseudo-Hawaiian earring subgroups.** We are now ready to prove:

Theorem 4.17 *The homeomorphism type of $B(X)$ is completely determined by $\pi_1(X, x_0)$, where $B(X)$ is the subspace of points at which X is not semilocally simply connected.*

Definition 4.4. We will say that x is a *wedge point* for $H \subset \pi_1(X, x_0)$ a pseudo-Hawaiian earring subgroup if, there exists a surjective homomorphism $\psi : \mathbb{H} \to H$ with the property that $\psi = \hat{\alpha} \circ f$ for a continuous function $f : \mathbb{E} \to X$ such that $f(0) = x$.

The following two lemmas are immediate from the definitions.

Lemma 4.5. *If X is a planar Peano continuum and $x \in B(X)$, then there exists H a pseudo-Hawaiian earring subgroup of $\pi_1(X, x_0)$ with x as its wedge point.*

Lemma 4.6. *If x is a wedge point for H a pseudo-Hawaiian earring subgroup, then x is a wedge point for $\widehat{\alpha}(H)$.*

We should show the existence and uniqueness of wedge points.

Lemma 4.7. *There exists a unique wedge point for each pseudo-Hawaiian earring subgroup.*

Proof. The existence follows Theorem 2.16.

Suppose the a pseudo-Hawaiian earring subgroup H has two distinct wedge points x and x' which correspond to two homomorphism ψ and ψ' with image H. Then there exists $f : (\mathbf{E}, 0) \to (X, x)$, $f : (\mathbf{E}, 0) \to (X, x')$, $\alpha : (I, 1) \to (X, x)$, and $\alpha' : (I, 1) \to (X, x')$ such that $\widehat{\alpha} \circ f_* = \psi$ and $\psi' = \widehat{\alpha}' \circ f'_*$.

Let U and U' be disjoint disks in the plane whose boundaries are not contained in X such that $U \cap X$ and $U' \cap X$ are neighborhoods of x and x' respectively. Since f and f' are continuous, the image of $f(\mathrm{a}_i)$ is eventually contained in $U \cap X$ and the image of $f'(\mathrm{a}_i)$ is eventually contained in $U' \cap X$. Hence there exists N such that $f_*(\mathbb{H}_N)$ is contained in $U \cap X$ and $f'_*(\mathbb{H}_N)$ is contained in $U' \cap X$

A countability argument shows that $\psi(\mathbb{H}_n) \cap \psi'(\mathbb{H}_n)$ is nontrivial for all n. However, by the previous paragraph any element in the intersection for $n > N$ can then be freely homotoped from U to U' which contradicts Lemma A4. $\qquad\square$

This lemma also implies that the definition of a wedge point is independent of our choice of homomorphism.

We will let $\mathcal{H}$ be the set of pseudo-Hawaiian earring subgroups of $\pi_1(X, x_0)$. Then we can now use the previous lemmas to define $\Upsilon : \mathcal{H} \to B(X)$, by taking a pseudo-Hawaiian earring subgroup to its wedge point. This function is surjective by Lemma 4.5. However, Lemma 4.6 implies that it is highly non-injective. We will now define an equivalence relation on $\mathcal{H}$ to make Υ injective on equivalency classes.

Definition 4.8. Let H_1, H_2 be pseudo-Hawaiian earring subgroups of $\pi_1(X, x_0)$. Then we will say that H_1 is *similar* to H_2 if there exists a $g \in \pi_1(X, x_0)$ and an $H \in \pi_1(X, x)$ a pseudo-Hawaiian earring subgroup such that $H_1, gH_2g^{-1} \subset H$. We will write $H_1 \sim H_2$.

If X is one-dimensional these equivalence classes are the same as maximal compatible nonempty subfamilies of the set of subgroups of $\pi_1(X, x_0)$ in [CE].

The following is well know.

Theorem 4.9. *For $\psi : \mathbb{H} \to \mathbb{F}$ a homomorphism where $\mathbb{F}$ is a free group, there exists an i such that ψ factors through P_{i*}.*

If we consider homomorphisms from the natural inverse limit containing $\mathbb{H}$ to free groups, then this is a theorem of Higman [H]. Morgan and Morrison also gave a topological proof in [MM]. When we consider homomorphism from $\mathbb{H}$, this is a result of Theorem 4.4 in [CC2] and a proof can be found in [S].

Lemma 4.10. *If x is a wedge point for H_1 and H_2, then $H_1 \sim H_2$.*

Proof. Let x be a wedge point for H_1 and H_2. Let $\psi_i : \mathbb{H} \to \pi_1(X, x_0)$ be a homomorphism with image H_i for $i \in \{1, 2\}$. Then there exists continuous functions $f_i : (\mathbf{E}, 0) \to (X, x)$ and $\alpha_i : (I, 1) \to (X, x)$ such that $\widehat{\alpha}_i \circ f_{i*} = \psi_i$.

Let $g : (\mathbf{E}, 0) \to (X, x)$ be sending $\mathbf{E}_o$ to $f_1(\mathbf{E})$ and $\mathbf{E}_e$ to $f_2(\mathbf{E})$, i.e. a_i goes to $f_1(a_{\frac{i+1}{2}})$ for odd i and $f_2(a_{\frac{i}{2}})$ for even i.

Hence $\widehat{\alpha}_i(H_i) \subset f_*(\mathbb{H})$ and $f_*(\mathbb{H})$ is a pseudo-Hawaiian earring subgroup. Then after conjugating H_2 by $\alpha_1 * \overline{\alpha_2}$ and $f_*(\mathbb{H})$ by α_1, we have necessary inclusion to show that $H_1 \sim H_2$. $\qquad\square$

Lemma 4.11. *If $H_1 \sim H_2$ and x is a wedge point for H_1, then x is a wedge point for H_2.*

Proof. It is sufficient to show that if H' and H are pseudo-Hawaiian earring subgroups with the property that $H' \subset H$, then H and H' have the same wedge point. However, this follows by just looking at a restriction of the continuous function which determines the wedge point of H. $\qquad\square$

Clearly, $\sim$ is reflexive and symmetic. Then Lemmas 4.10 and 4.11 show that $\sim$ is transitive. Hence, $\sim$ is actually an equivalency relation. We will use $[H]$ to denote the equivalency class of pseudo-Hawaiian earring subgroups generated by the equivalency relation $\sim$.

We can now give an equivalent definition for the wedge point of a psuedo-Hawaiian earring subgroup. Let $H = \langle\langle g_n \rangle\rangle$ be a psuedo-Hawaiian earring subgroup. Then the wedge point of H is the unique point, x, which enjoys the following property: *For every $\epsilon > 0$, there exists an N such that the set of homotopy classes $\{g_n\}_{n>N}$ have representatives contained in $B_\epsilon(x)$.*

Proposition 4.12. The set $B(X)$ corresponds bijectively to equivalence classes of pseudo-Hawaiian earring subgroups of $\pi_1(X, x_0)$, i.e. $\Upsilon^{-1}(x) = [H]$

Then it makes sense to talk about the *wedge point of a pseudo-Hawaiian earring subgroup equivalency class* being the wedge point of any of its members.

Definition 4.13. Let $\{[H_n]\}$ be a sequence of equivalence classes of pseudo-Hawaiian earring subgroups of the fundamental group of a planar Peano continuum. We will say that $[H_n]$ *converges* to $[H]$, a pseudo-Hawaiian earring subgroup equivalence class, if there exists a choice of representatives $\langle\langle g_{n,i}\rangle\rangle_i \in [H_n]$ and a sequence of natural numbers $\{M_n\}$ so that for any sequence $\{k_n\}$, with $k_n > M_n$ the pseudo-Hawaiian earring subgroup $\langle\langle g_{n,k_n}\rangle\rangle_n \in [H]$. We will use all the standard terminology when referring to sequence of pseudo-Hawaiian earring subgroup equivalence classes.

Actually since we have to be able to choose the representative for the equivalence class, we may as well choose the representative such that $M_n = 1$. This can be done by removing the first finitely many big loops which will not change the equivalence class.

Initially there is no reason to suppose that a convergent sequence has a unique limit point. However, this will be a consequence of Proposition 4.14 and Lemma 4.10.

Proposition 4.14. If $[H_n] \to [H]$, then $x_n \to x$ where x_n is the wedge point of $[H_n]$ and x is the wedge point of $[H]$.

Proof. Suppose that $[H_n]$ *converges* to $[H]$ a pseudo-Hawaiian earring subgroup equivalence class. Then there exists a choice of representatives $\langle\langle g_{n,i}\rangle\rangle_i \in [H_n]$ and $\{M_n\}$ such that if $k_n \geq M_n$, then $\langle\langle g_{n,k_n}\rangle\rangle_n \in [H]$.

We will proceed by way of contradiction. Let x_n be be the wedge point for $[H_n]$. Suppose that $x_n \nrightarrow x$. Then there is a subsequence x_{n_i} with the property that $x_{n_i} \to x' \neq x$. Fix A, A' disjoint open disk neighborhoods of x and x' respectively, whose boundaries are not contained in X. Then we may choose $\{M'_{n_i}\}_i$ and N' such that g_{n_i,k_i} can be freely homotoped into A' for $n_i > N'$ and $k_i > M'_{n_i}$. Let $M'_k = 0$ if $k \neq n_i$ for any i.

Fix $g_{n,k(n)}$ such that $k(n) > \max\{M'_n, M_n\}$. Then $\langle\langle g_{n,k(n)}\rangle\rangle_n \in [H]$ and there exists an N such that, for $n > N$, $g_{n,k(n)}$ can be freely homotoped into A. Then for $n_i > \max\{N', N\}$ $g_{n_i,k(n_i)}$ can be freely homotoped into A and A', a contradiction.

$\square$

The uniqueness of the limit of a convergent sequence then follows by considering the corresponding sequence of wedge points.

Let $\overline{\mathcal{H}}$ be the set of equivalence classes of pseudo-Hawaiian earring subgroups. We will now construct a topology on $\overline{\mathcal{H}}$.

Definition 4.15. We will say that $A \subset \overline{\mathcal{H}}$ is *closed* if it contains all of its limit points. Then a set is *open* in $\overline{\mathcal{H}}$ if its complement is closed.

It is an easy exercise to show that this defines a topology on $\overline{\mathcal{H}}$.

Let $\overline{\Upsilon} : \overline{\mathcal{H}} \to B(X)$ by $\overline{\Upsilon}([H]) = \Upsilon(H)$. This is well defined by Lemma 4.11 and a bijection by Proposition 4.12.

Proposition 4.14 is exactly what is required for $\overline{\Upsilon}$ to be continuous. Suppose $C \subset B(X)$ which is closed. Consider $\{[H_n]\} \subset \overline{\Upsilon}^{-1}(C)$ which converges to $[H]$. If x_n is the wedge point of $[H_n]$, then x_n is in C and x_n must converge to x the wedge point of $[H]$. Thus $x \in C$ and $[H] \in \overline{\Upsilon}^{-1}(C)$.

The next lemma will give the continuity of $\overline{\Upsilon}^{-1}$ immediately since X is metric.

Lemma 4.16. *If $x_n \in B(X)$ and and $x_n \to x$, then the corresponding pseudo-Hawaiian Earring subgroup equivalency classes converge.*

Proof. Let H_n be a pseudo-Hawaiian Earring subgroup of $\pi_1(X, x_0)$ with wedge point x_n. Then there exists a continuous function $f_n : (\mathbf{E}, 0) \to (X, x_n)$ which induces an injective homomorphism. There exists a path $T : I \to X$ with the property that $T(0) = x$, $T(1) = x_1$, and $T(\frac{1}{n}) = x_n$. Let $\alpha : I \to X$ be any path from x to x_0. Let β_n be a parametrization of the path $\alpha * T\big|_{[0, \frac{1}{n}]}$. Then $\langle\langle \overline{\beta_n} * f_n(\mathrm{a}_i) * \beta_n \rangle\rangle_i$ has wedge point x_n. Let $g_{n,i} = \overline{\beta_n} * f_n(\mathrm{a}_i) * \beta_n$. There exists M_n such that $f_n(\mathrm{a}_i) \subset B_{\frac{1}{n}}(x_n)$ for $i > M_N$. If $k \geq M_n$ for each n, then $\langle\langle g_{n,k} \rangle\rangle_n \in [H]$. $\qquad\square$

Therefore we have know proved the following result.

Theorem 4.17. *The function $\overline{\Upsilon} : \overline{\mathcal{H}} \to B(X)$ is a homeomorphism, i.e. the homeomorphism type of $B(X)$ is completely determined by $\pi_1(X, x_0)$.*

APPENDIX

Greg Conner and Mark Meilstrup proved the following lemma in [CM].

Lemma A1. *Let H be a function from the first-countable space $X \times Y$ into Z. Let $\{C_i\}$ be a null sequence of closed sets whose union is X. Suppose that $\{D_i = H(C_i \times Y)\}$ is a null sequence of sets in Z and H is continuous on each $C_i \times Y$. If for every subsequence $C_{i_k} \to x_0$ there exists a $z_0 \in Z$ such that $D_{i_k} \to z_0$ then H is continuous on $X \times Y$.*

Proof. Consider a sequence $(x_n, y_n) \to (x_0, y_0)$. For each n, choose an i_n such that $x_n \in C_{i_N}$. If $\{C_{i_n}\}$ is finite then by restricting H to $\cup_n C_{i_n} \times Y$ we have $H(x_n, y_n) \to H(x_0, y_0)$ be a finite application of the pasting lemma. If $\{C_{i_n}\}$ is infinite, then since $\{C_i\}$ is a null sequence and $x_n \in C_{n_i}$, we have $C_{i_n} \to x_0$ and thus $H(x_n, y_n) \in D_{i_n} \to z_0 = H(x_0, y_o)$. Thus H is continuous on all of $X \times Y$. $\qquad\square$

Lemma A2. *If A is an annulus and C is a closed subset of $\mathrm{int}\, A$ that separates the boundary components J_1 and J_2 of A, then some component of C separates J_1 from J_2 in A.*

This is a consequence of Phragmén-Brouwer properties. (see [HW].)

Lemma A3. *If X is a planar Peano continuum which is not semilocally simple connected at x, then there exists arbitrarily small disks neighborhoods of x in $\mathbb{R}^2$ which enjoy the property that their boundary is not contained in X.*

Proof. Consider $B_\epsilon(x) \subset \mathbb{R}^2$. If $\partial(B_\epsilon(x)) \subset \mathbb{R}^2$ is contained in X for all sufficiently small ϵ, then x has a simply connected neighborhood. Hence X is simply connected at x. $\square$

Lemma A4. *Let X be a subset of $\mathbb{R}^2$ and U a closed disk in the plane whose boundary is not contained in X. If α is an essential loop contained in $X \cap \mathrm{int}(U)$, then α cannot be freely homotoped out of $U \cap X$.*

Proof. Let X be a subset of $\mathbb{R}^2$ and U a closed disk in the plane whose boundary is not contained in X. Suppose α is an essential loop contained in $X \cap \mathrm{int}(U)$ such that α can be homotoped out of $U \cap X$.

Let A be an annulus with boundary components J_1 and J_2. Then there exists a map $h : A \to X$ which takes J_1 to α and J_2 to a loop in X not intersecting $U \cap X$. Then $h^{-1}(\partial U \cap X)$ is a closed subset contained in the $\mathrm{int}\, A$ which separates J_1 from J_2 in A. Hence by Lemma A2, some component C of $h^{-1}(\partial U \cap X)$ separates J_1 from J_2 in A. Since $\partial U \not\subset X$, $h\big|_C$ maps to an arc in X. Hence by Tietze Extension Theorem, we may alter h to a map of a disk into X. Therefore α is nullhomotopic, a clear contradiction.

$\square$

Lemma A5. *If $x_n \to x$ in a Peano continuum, then there exists a path $T : I \to X$ with the property that $T(0) = x_1$, $T(1) = x$, and $T(1 - \frac{1}{n}) = x_n$.*

Proof. Let U_n be a path connected neighborhood of x contained in $B_{\frac{1}{n}}(x) \cap U_{n-1}$. Let $T_n : [1 - \frac{1}{n}, 1 - \frac{1}{n+1}] \to X$ be any path from x_n to x_{n+1} with the property that if $x_n, x_{n+1} \in U_k$ then $T_n([1 - \frac{1}{n}, 1 - \frac{1}{n+1}]) \subset U_k$. T_n will exist since there is a minimal U_k containing both x_n and x_{n+1}. Then $T : I \to X$ by $T(x) = T_n(x)$ for $x \in [1 - \frac{1}{n}, 1 - \frac{1}{n+1}]$ and $T(1) = x$ is a continuous path. $\square$

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